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MULTIPERIOD SECURITIES MARKETS WITH DIFFERENTIAL INFORMATION: MARTINGALES AND RESOLUTION TIMES*

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Abstract

We model multiperiod securities markets with differential information. A price system that admits no free lunches is related to martingales when agents have rational expectations. We introduce a concept called *resolution time* and show that a better informed agent and a worse informed agent must agree on the resolution times of commonly marketed events if they have rational expectations and if there are no free lunches. It then follows that if all the elementary events are marketed for a worse informed agent then any price system that admits no free lunches to a better informed agent must dynamically equalize the information asymmetry between the two. We provide an example of a dynamically fully revealing price system that is arbitrage free and yields elementarily complete markets.

1. Introduction and summary

This paper addresses differential information in a multiperiod model of security markets. Focusing on asset price processes that preclude "free lunches," we first extend the famous Harrison–Kreps connection between price processes and martingales. We go on to study the information that must be revealed by a better informed agent through price processes to a worse informed agent if free lunches are to be precluded. Finally, we formalize the connection between completely revealing price processes and dynamically complete markets for elementary contingent claims. The details and additional results are summarized below. The results of Harrison and Kreps [1979] connecting the behavior of price processes with martingales have already opened up a theory of dynamic equilibrium with symmetric information (Duffie and Huang [1985]). The results here are a step toward dynamic equilibrium with differential information.

The paper is summarized as follows. Section 2 motivates the concepts of differential information in a continuous-time setting. The less demanding discrete-time details can easily be surmized by readers. Section 3 presents the formal model and shows that, in a rational expectations model, agents agree on the joint variation of stochastic price processes.

A contingent claim is *marketed* if financed by some security trading strategy. Section 4 shows some of the advantages of better information: a larger space of admisible trading strategies and thus a larger space of marketed claims. Section 5 shows that the absence of *simple free lunches* (Kreps [1981]) implies a unique implicit price process for any marketed claim. A better informed agent cannot, in the absence of simple free lunches, attain a consumption claim at a smaller initial investment than a worse informed agent. Kreps [1981] shows a need to strengthen the *simple free lunch* concept of "no arbitrage" to the *free lunch* in infinite-dimensional cases. Barring free lunches for a particular agent, we demonstrate the existence of a *martingale measure* for that agent: a probability measure, absolutely continuous with respect to the agent's endowed probability measure, under which security price processes are martingales.

In Section 6 we introduce the *resolution time* of an event, the first time one learns that an event is to happen or not to happen with probability one. The resolution time of an event is random, and of course may be different for differently informed agents. A better informed agent has no later resolution time than a worse informed agent. An event B is *marketed* for an agent if there exists a marketed elementary contingent claim for this event, or equivalently, if there exists a trading strategy paying one unit of consumption in event B and nothing otherwise. By assuming the consumption space is separable and thereby extending a result of Kreps [1981], we are able to demonstrate the existence of an *equivalent martingale measure*, barring free lunches. This is a martingale measure assigning strictly positive probability to any event assigned non–zero probability by the agent's endowed probability measure. This allows us to show that the absence of free lunches for the better informed agent implies that the resolution times of the better and worse

informed agents for commonly marketed events must be equal (almost surely). It follows that, if all events are marketed for the poorly informed agent and if the better informed agent has no free lunches, then the price system has symmetrized the information, neutralizing the better informed agent's informational advantage.

Section 7 provides an example of a fully revealing arbitrage-free price system. Uncertainty is modeled by a Brownian Motion. Agent α observes the Brownian Motion directly. Agent β does not observe the Brownian Motion as it evolves through time, but knows even at the beginning of time the final value of the Brownian Motion. Two long-lived securities are traded, one risky and one riskless. The price process of the risky security symmetrizes the original informational assymetry between agents α and β. Markets are dynamically complete for elementary claims. Section 8 adds some discussion, generalizations, and a brief concluding remark.

2. Differential Information

The central primitive for a model of uncertainty is a set Ω of "states of the world," one of which is "correct," loosely speaking. An agent, say Fred, receives information through time, refining Fred's knowledge concerning which of the states in Ω is correct. For this section the *time set* $\mathbf{T}$ is any ordered subset of $[0,1]$, allowing one to cover continuous and discrete time, in a single pass. How Fred receives information is formalized by specifying a *filtration* $\mathbf{F} = \{\mathcal{F}_t; t \in \mathbf{T}\}$ of *tribes* (also termed σ–algebras) on Ω. If some subset A of Ω is an element of the tribe $\mathcal{F}_t$ then at time t Fred knows whether or not the correct state is an element of A, that is, whether A has "occurred". To formalize the notion that such an occurrence is never forgotten, we say that $\mathbf{F}$ is *increasing*, or

$$\mathcal{F}_t \subset \mathcal{F}_s \qquad \forall s \geq t.$$

A further assumption which is often technically convenient when $\mathbf{T}$ is a continuum time set is that the filtration $\mathbf{F}$ is *right–continuous*, or

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s.$$

A different agent, say George, might naturally receive different information through time specified by a filtration $\mathbf{G} = \{\mathcal{G}_t; t \in \mathbf{T}\}$ of tribes on Ω. Of course, if $\mathcal{G}_t \subset \mathcal{F}_t$ then George is at an informational disadvantage at time t, since Fred "knows" every event that George knows at time t, and perhaps more. If $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t \in \mathbf{T}$, we write $\mathbf{G} \subset \mathbf{F}$ and say that Fred is *better informed* than George.

The *join* of $\mathcal{F}_t$ and $\mathcal{G}_t$, denoted $\mathcal{F}_t \vee \mathcal{G}_t$, is the tribe generated by the union of $\mathcal{F}_t$ and $\mathcal{G}_t$, and represents the total information held by the two agents at time t. In other words, if $B \in \mathcal{F}_t \vee \mathcal{G}_t$ then at time t, by pooling their information, Fred and George would know whether B has occurred. The joined filtration, denoted $\mathbf{F} \vee \mathbf{G}$ is, of course $\{\mathcal{F}_t \vee \mathcal{G}_t; t \in \mathbf{T}\}$.

The *meet* of $\mathcal{F}_t$ and $\mathcal{G}_t$, on the other hand, is the intersection of $\mathcal{F}_t$ and $\mathcal{G}_t$ (which is indeed a σ–algebra), and is denoted $\mathcal{F}_t \wedge \mathcal{G}_t$. If $B \in \mathcal{F}_t \wedge \mathcal{G}_t$ then Fred and George know independently at time t whether B has occurred. The *meet* of the filtrations is denoted $\mathbf{F} \wedge \mathbf{G} = \{\mathcal{F}_t \wedge \mathcal{G}_t; t \in \mathbf{T}\}$.

In the "rational expectations" genre it is common to assume that agents have private information and also learn information from "market observables", in particular, the market values of traded assets. Market observables can be represented as a vector of *stochastic processes*, which we take to be nothing more than function S from $\Omega \times \mathbf{T}$ to $\Re^N$ for some integer N. The value of S at time t is denoted $S(t)$, a function on Ω. The information revealed by observing S up to and including time t is modeled as the tribe $\mathcal{F}_t^S$ on Ω generated by the functions $\{S(s); s \in [0,t]\}$. The corresponding filtration of tribes generated by S is denoted $\mathbf{F}^S = \{\mathcal{F}_t^S; t \in \mathbf{T}\}$. To belabor the point, if $B \in \mathcal{F}_t^S$ then one will know at time t whether or not B has occurred by observing the process S up to and including time t.

If an agent is endowed with private information corresponding to a filtration $\mathbf{G}$ and learns from a vector of market observables S, then the agent's "total" information is of course $\mathbf{G} \vee \mathbf{F}^S$.

Suppose that agents are from a set $\mathcal{A}$, indexed by α, with private information given by filtrations $\mathbf{F}^\alpha$. The market observable process S is *fully revealing* if

$$\bigvee_\alpha \mathbf{F}^\alpha \subset \mathbf{F}^\alpha \vee \mathbf{F}^S \qquad \forall \alpha \in \mathcal{A},$$

which means that by observing S each agent learns all privately held information not already known.

A stochastic process S is *adapted* to a filtration $\mathbf{G} = \{\mathcal{G}_t; t \in \mathbf{T}\}$ if $S(t)$ is measurable with respect to $\mathcal{G}_t$ for all $t \in \mathbf{T}$. Roughly, S is adapted to $\mathbf{G}$ if S only conveys information already implicit in $\mathbf{G}$. Indeed, S is adapted to $\mathbf{G}$ if and only if $\mathbf{F}^S \subset \mathbf{G}$.

The question arises: Are market observables such as prices naturally adapted to the join of all private information? This is essentially a question about the economic structure of price formation. If one believes that market values are a direct consequence of individual strategies, such as bidding, for example, then it would be natural to treat market observables as adapted. One could also imagine, perhaps, that certain market structures add "noise" to price formation. This issue is also addressed in Kreps [1977]. For our purposes, we are not forced to take a stance.

3. The formulation

In this section we consider an intertemporal pure exchange economy under uncertainty with differential information. Taken as primitive is a complete probability space $(\Omega, \mathcal{F}, P)$, where each $\omega \in \Omega$ denotes a complete description of the exogenous uncertain environment, and where P is the common probability assessment held by agents in the economy. The time set $\mathbf{T}$ is taken to be a subset of the interval $[0,1]$, which includes 0 and 1.

It is assumed that there is one consumption good in the economy which is available for consumption only at time 1. We take $V \equiv L^1(P)$ to be the commodity space, where $L^1(P)$ denotes the space of integrable random variables on $(\Omega, \mathcal{F}, P)$. Thus $v \in V$ means $v(\omega)$ units of consumption at time one in state $\omega \in \Omega$. We equip V with the L^1-norm topology denoted by τ.

An *information structure* is an increasing family of subtribes of $\mathcal{F}$, $\{\mathcal{F}_t; t \in \mathbf{T}\}$, that is, a *filtration*. Let $\widehat{\mathbf{F}}$ be a countable collection of information structures.

The economy is populated with agents from a set $\mathcal{A} = \{\succeq_\alpha, \mathcal{V}_\alpha, \mathbf{F}^\alpha; \alpha \in A\}$. We shall assume that the following conditions hold for each $(\succeq_\alpha, \mathcal{V}_\alpha, \mathbf{F}^\alpha) \in \mathcal{A}$:

1 $\mathcal{V}_\alpha \subset V$;

2 $\succeq_\alpha$ is a preference relation defined on $\mathcal{V}_\alpha$ that is strictly increasing in the sense that $v \in \mathcal{V}$ and $k \in K$ implies that $v + k \succ_\alpha v$, where $\succ_\alpha$ is the strict preference relation derived from $\succeq_\alpha$, and where K is the positive cone of V with the origin deleted;

3 $\mathbf{F}^\alpha \in \widehat{\mathbf{F}}$.

We interpret $\mathcal{V}_\alpha$ to be agent α's net trade space, $\succeq_\alpha$ to be his preference relation on $\mathcal{V}_\alpha$, and $\mathbf{F}^\alpha$ to be his endowed information structure.

It is assumed that there are a finite number of long-lived securities traded in the economy in zero net supply indexed by $n = 1, 2, \ldots, N$. Each security n is represented by $d_n \in K$. We assume that $\sum_{n=1}^N d_n = 1$ *a.s.*, a normalization. We shall henceforth denote $(d_n)_{n=1}^N$ by d. To rule out degeneracy, we further impose that for all n, $P\{d_n < 1\} = 1$.

A price system for long-lived securities is an N-vector of nonnegative processes $S = \{S_n(t); n = 1, 2, \ldots, N; t \in \mathbf{T}\}$ satisfying the following conditions:

1 S is a vector of semimartingales with respect to some information structure $\mathbf{F}$ such that $\mathbf{F}^\alpha \subset \mathbf{F}$ for all $\alpha \in A$;[1]

2

$$S_n(t) < 1 \quad a.s. \text{ and } \sum_{n=1}^N S_n(t) = 1 \quad a.s. \forall t \in \mathbf{T}; \tag{3.1}$$

3 for $n = 1, 2, \ldots, N$,

$$E\left({}^{\mathbf{F}}[S_n, S_n]_1\right)^{\frac{1}{2}} < \infty, \tag{3.2}$$

where $\{{}^{\mathbf{F}}[S_n, S_n]_t\}$ is the *joint variation* process of S_n and S_n with respect to $\mathbf{F}$; cf. Dellacherie and Meyer [1982, VII.44].

We assume in addition that $\mathbf{F}$ is *complete* in that $\mathcal{F}_t$ contains all the P-null sets for all $t \in \mathbf{T}$.

Requiring S to be a semimartingale with respect to some information structure $\mathbf{F}$ is without loss of generality. Any discrete time process is a semimartingale with respect to its natural filtration.

[1] For the definition of a semimartingale see, for example, Jacod [1979, pp.29].

In the continuous time case, almost all the known processes are semimartingales (with respect to some filtration). The assumption that $\mathbf{F}$ is finer than any of $\{\mathbf{F}^\alpha; \alpha \in A\}$ allows the possibility that an agent can learn from a price system. There exists discussions in the literature, for example, Kreps [1977], on whether $\mathbf{F}$ should be at most as fine as the joint of all agents' information. This is not an issue that concerns us. Our assumption is general enough to include that as a special case. Condition (3.1) is again a normalization and condition (3.2) is a technical restriction.

Remark 3.1: For a sufficient condition for (3.2) see Dellacherie and Meyer [1982, VII.98].

Given a price system S, the *filtration* it generates is denoted by $\mathbf{F}^S = \{\mathcal{F}_t^S; t \in \mathbf{T}\}$.

It is assumed that agents in $\mathcal{A}$ have *rational expectations* in that they learn from a price system to refine their information. The information structure of agent α after he observes a price system S is denoted by $H^\alpha(S) = \{h_t^\alpha(S); t \in \mathbf{T}\}$, where

$$h_t^\alpha(S) = \mathcal{F}_t^\alpha \bigvee \mathcal{F}_t^S.$$

We shall assume throughout that $H^\alpha(S)$ satisfies the *usual conditions*:

1 *complete*; and

2 *right-continuous*: let $[r, s)$ be any half open interval contained in $\mathbf{T}$, then $\forall t \in [r, s)$, $\mathcal{F}_t = \bigwedge_{u>t} \mathcal{F}_u$, where $u \in [r, s)$.

For consistency, we need to require also that d be measurable with respect to $h_1^\alpha(S)$ for all $\alpha \in A$. That is, at time 1 when agents' long and short positions of long-lived securities need to be settled, they will know how many units of consumption good to which they are entitled or obligated.

Before proceeding, we shall first record some technical results.

Lemma 3.1. *A price system is an N-vector of $H^\alpha(S)$-semimartingales for all α.*

Proof. We first claim that S is $H^\alpha(S)$-optional. By construction, S is adapted to $H^\alpha(S)$ and is right continuous, therefore $H^\alpha(S)$-optional; cf. Chung and Williams [1983, Theorem 3.4]. Thus S is an $\mathbf{F}$-semimartingale and is $H^\alpha(S)$-optional. It follows then from Theorem 9.19(a) of Jacod [1979] that S is an $H^\alpha(S)$-semimartingale, since $H^\alpha(S)$ is a *sub-filtration* of $\mathbf{F}$.

∎

If $\mathbf{F}^S$ is not coarser than $\mathbf{F}^\alpha$ and agent α ignors the information revealed through S, the price system S will *not* be an $\mathbf{F}^\alpha$-semimartingale, since it is not adapted to $\mathbf{F}^\alpha$. In an intertemporal economy, where trading over time is allowed, before anything interesting can be said, an intertemporal budget constraint for agents must be formulated. With uncertainty present, this budget constraint must involve stochastic integration. Jacod [1979, p. 278-279] has shown that in order for stochastic integrals to have desired properties, it is *necessary* that the integrators be semimartingales. Thus

when a price system carries information with which an agent is not endowed, he must learn it. Or else, the agent's intertemporal budget constraint is not well defined.

Another lemma is needed:

Lemma 3.2. *There exists a process* $\{[S_n, S_m]_t; t \in \mathbf{T}\}$ *which is a common version* [2] *of the processes* $\{^{\mathbf{F}}[S_n, S_m]_t; t \in \mathbf{T}\}$ *and* $\{^{H^\alpha(S)}[S_n, S_m]_t; t \in \mathbf{T}\}$, *where* $\{^{H^\alpha(S)}[S_n, S_m]_t; t \in \mathbf{T}\}$ *is the joint variation process of* S_n *and* S_m *with respect to* $H^\alpha(S)$.

Proof. Since S_n and S_m are semimartingales with respect to $\mathbf{F}$ and $H^\alpha(S)$, the two processes $\{^{\mathbf{F}}[S_n, S_m]_t; t \in \mathbf{T}\}$ and $\{^{H^\alpha(S)}[S_n, S_m]_t; t \in \mathbf{T}\}$ are well defined; cf. Dellacherie and Meyer [1982, VII.42]. We first note that

$$^{\mathbf{F}}[S_n, S_m]_t = \frac{1}{4}\left\{^{\mathbf{F}}[S_n + S_m, S_n + S_m]_t - ^{\mathbf{F}}[S_n - S_m, S_n - S_m]_t\right\},$$

and

$$^{H^\alpha(S)}[S_n, S_m]_t = \frac{1}{4}\left\{^{H^\alpha(S)}[S_n + S_m, S_n + S_m]_t - ^{H^\alpha(S)}[S_n - S_m, S_n - S_m]_t\right\};$$

cf. Jacod [1979], Section 2.25. Theorem 9.19(b) of Jacod [1979] shows that there exists a common version of $\{^{\mathbf{F}}[S_n + S_m, S_n + S_m]_t\}$ and $\{^{H^\alpha(S)}[S_n + S_m, S_n + S_m]_t\}$, and similarly for $\{^{\mathbf{F}}[S_n - S_m, S_n - S_m]_t\}$ and $\{^{H^\alpha(S)}[S_n - S_m, S_n - S_m]_t\}$. Therefore there esists a common version of $\{^{\mathbf{F}}[S_n, S_m]_t\}$ and $\{^{H^\alpha(S)}[S_n, S_m]_t\}$.

∎

We shall henceforth use $\{[S_n, S_m]_t\}$ to denote a common version of $\{^{\mathbf{F}}[S_n, S_m]_t\}$ and $\{^{H^\alpha(S)}[S_n, S_m]_t\}$ for all α. (This is possible because $\widehat{\mathbf{F}}$ is a countable set.) Note that $\{[S_n, S_m]_t\}$ is adapted to both $\mathbf{F}$ and $H^\alpha(S)$ since both filtrations are complete. In addition, since joint variation processes are right continuous, $\{[S_n, S_m]_t\}$, $\{^{\mathbf{F}}[S_n, S_m]_t\}$, and $\{^{H^\alpha(S)}[S_n, S_m]_t\}$ are *indistinguishable*[3] processes. Indistinguishable processes are indeed the same process for any pratical purpose.

Lemma 3.2 has an important implication. As long as agents have rational expectations, they will agree on the joint variation processes of security prices. In the case that price processes are continuous processes, joint variation processes are covariance processes. Thus rational agents must agree on the variance-covariance matrix process.

Now let $\mathcal{P}_\alpha(S)$ be the tribe of subsets of $\Omega \times [0,1]$ generated by left continuous $H^\alpha(S)$-adapted processes. A process $Y : \Omega \times [0,1] \mapsto \Re$ is said to be $H^\alpha(S)$-*predictable* if it is measurable with respect to $\mathcal{P}_\alpha(S)$.

Given a price system S, an admissible trading strategy for agent α is an N-vector of processes $\theta = \{\theta_n(t); n = 1, 2, \ldots, N; t \in \mathbf{T}\}$ satisfying the following conditions:

[2] A process $\{X(t)\}$ is a version of another process $\{Y(t)\}$ if $X(t) = Y(t)$ with probability one for all $t \in \mathbf{T}$.
[3] Two processe $\{X(t)\}$ and $\{Y(t)\}$ are indistinguishable if $X(t) = Y(t)$ for all $t \in \mathbf{T}$ with probability one.

1 θ is $H^\alpha(S)$-predictable;

2 the stochastic integral

$$\int_0^t \theta(s)^\top dS(s)$$

is well defined with respect to $H^\alpha(S)$ for all $t \in \mathbf{T}$, henceforth denoted by $\theta \overset{H^\alpha(S)}{\cdot} S$;

3 θ is *self-financing*, that is,

$$\theta(t)^\top S(t) = \theta(0)^\top S(0) + \int_0^t \theta(s)^\top dS(s) \qquad \forall t \in \mathbf{T} \quad a.s. \tag{3.3}$$

with respect to $H^\alpha(S)$, and $\theta(1)^\top d \in V$;

4 for all $n = 1, 2, \ldots, N$

$$E\left(\int_0^1 (\theta_n(t))^2 \, d[S_n, S_n]_t\right)^{\frac{1}{2}} < \infty. \tag{3.4}$$

Condition (3.3) is a natural budget constraint and condition (3.4) is technical.

Let $\Theta^\alpha[S]$ denote the space of admissible trading strategies for agent α when the price system is S. By the linearity of the stochastic integral and an application of the Kunita-Watanabe inequality (cf. Dellacherie and Meyer [1982, pp.277]), we know $\Theta^\alpha[S]$ is a linear space.

4. The advantage of better information

We shall show in this section that a better informed agent will be better off in the sense that he or she has access to more admissible trading strategies and therefore can enjoy a larger feasible net trade space. The direct converse of the above statement is not true, however. We show that if agent α has access to a bigger space of admissible strategies than agent β, say. Then agent α's information as refined by a price system must be finer than that of agent β except possibly at date 1.

Following Harrison and Kreps [1979] we say that a consumption claim $v \in V$ is marketed for agent α given a price system S if there exists $\theta \in \Theta^\alpha[S]$ such that $\theta^\alpha(1)^\top d = v$ *a.s.* In such event, θ is said to generate v for agent α and $\theta(0)^\top S(0)$ is an implicit price for v at time zero. Let M_α denote the space of marketed consumption claims for agent α. By the fact that $\Theta^\alpha[S]$ is a linear space, M_α is a linear subspace of V.

We shall say that agent α is better informed than agent β if $\mathbf{F}^\beta$ is a *sub-filtration* of $\mathbf{F}^\alpha$, that is, $\mathcal{F}_t^\beta \subset \mathcal{F}_t^\alpha \; \forall t \in \mathbf{T}$.

Theorem 4.1. *If agent α is better informed than agent β, then $\Theta^\beta[S] \subset \Theta^\alpha[S]$ and therefore $M_\beta \subset M_\alpha$.*

Proof. Let $\theta \in \Theta^\beta[S]$. We want to show that θ satisfies the defining properties of $\Theta^\alpha[S]$.

Firstly, the fact that the stochastic integral $\theta \overset{H^\alpha(S)}{\cdot} S$ is well defined follows from the first assertion of Theorem 9.26 of Jacod [1979]. Also, $\theta(1)^\top d \in V$ since $\theta \in \Theta^\beta[S]$.

Secondly, the fact that θ is a self-financing strategy with respect to $H^\alpha(S)$ follows from the second assertion of Theorem 9.26 of Jacod [1979].

Lastly, the fact that θ satisfies (3.4) is obvious. Therefore $\theta \in \Theta^\alpha[S]$ and thus $M_\beta \subset M_\alpha$.

∎

As long as a better informed agent has rational expectations, whatever strategy a worse informed agent can employ can be employed by him. The converse of the above proposition is incorrect, however. Consider the following example. Let α and β be two agents, whose information structures cannot be ordered. Suppose that the information generated by the price system is identical to $\mathbf{F}^\beta$. Therefore $H^\beta(S) \subset H^\alpha(S)$ and thus $\Theta^\beta[S] \subset \Theta^\alpha[S]$.

The following proposition shows that the statement that $\Theta^\beta[S] \subset \Theta^\alpha[S]$ implies $H^\beta[S] \subset H^\alpha[S]$ is also incorrect.

Proposition 4.1. *Suppose that* $h_t^\beta(S) \subset h_t^\alpha(S)\ \forall t \in \mathbf{T}/\{0\}$. *Then* $\mathcal{P}_\beta(S) \subset \mathcal{P}_\alpha(S)$ *and* $\Theta^\beta[S] \subset \Theta^\alpha[S]$.

Proof. The predictable tribe $\mathcal{P}_\beta(S)$ is generated by the collection of $H^\beta(S)$-*predictable rectangles* of the form $\{0\} \times B_0$ and $(s,t] \times B$ with $B_0 \in h_0^\beta(S)$ and $B \in h_s^\beta(S)$ for $s,t \in \mathbf{T}$, $s < t$; cf. Chapter 2 of Chung and Williams [1983]. By the hypothesis that $\mathcal{F}_t^\beta \subset \mathcal{F}_t^\alpha\ \forall t \in \mathbf{T}/\{0\}$, we know that every $\mathcal{P}_\beta(S)$-predictable rectangle is a $\mathcal{P}_\alpha(S)$-predictable rectangle. Therefore, $\mathcal{P}_\beta(S) \subset \mathcal{P}_\alpha(S)$. The rest of the assertion follows from similar arguments as those of Theorem 4.1.

∎

The intuition behind Proposition 4.1 is clear. In a finite horizon single commodity economy where admissible strategies are predictable ones, agents are not able to take advantage of their better information revealed to them at the final date of the economy. There will be no trading at the final date! A correct statement is as follows.

Theorem 4.2. *Suppose that* $h_1^\beta(S) \subset h_1^\alpha(S)$. *Then* $\Theta^\beta[S] \subset \Theta^\alpha[S]$ *implies* $H^\beta(S) \subset H^\alpha(S)$.

Proof. Suppose that there exists $t \in \mathbf{T}/\{1\}$ such that $h_t^\beta(S) \not\subset h_t^\alpha(S)$. Then there is a set $B \in h_t^\beta(S)$ and $B \notin h_t^\alpha(S)$. The rectangle $(t,1] \times B$ is an element of $\mathcal{P}_\beta(S)$ but is not an element of $\mathcal{P}_\alpha(S)$; cf. Chung and Williams [1983, pp. 28]. Define a self-financing trading strategy by

$$\theta_i(\omega,s) = 1_{(t,1]\times B}(\omega,s)\,(1 - S_i(\omega,t)) \text{ for some } i$$
$$\theta_n(\omega,s) = -1_{(t,1]\times B}(\omega,s) S_i(\omega,t) \text{ for all } n \neq i.$$

It is quickly checked that θ defined above is an element of $\Theta^\beta[S]$. But θ is not $\mathcal{P}_\alpha(S)$-predictable. To see this we note that, $\theta_i^{-1}(\Re_+/\{0\}) = (t,1] \times B \notin \mathcal{P}_\alpha(S)$. This contradicts the hypothesis that $\Theta^\beta[S] \subset \Theta^\alpha[S]$. Thus we must have $H^\beta(S) \subset H^\alpha(S)$.

∎

A corollary is available.

Corollary 4.1. *Suppose that* $h_1^\alpha(S) = h_1^\beta(S)$ *and that* $\Theta^\alpha[S] = \Theta^\beta[S]$. *Then* $H^\alpha(S) = H^\beta(S)$.

Proof. Using Theorem 4.2 twice gives the result.

∎

It is clear from the above discussion that whether a converse statement of Theorem 4.1 is true will depend upon whether the information conveyed by a price system is a *substitute* or a *complement* to agents' information.

5. Free lunches and martingales

In this section we shall show that a price system that admits no free lunches is intimately connected with martingales when agents have rational expectations. The definitions of a *simple free lunch* and a *free lunch* that we employ are those of Kreps [1981].

Definition 5.1. A simple free lunch for agent α is a strategy $\theta \in \Theta^\alpha[S]$ such that $\theta(0)^\top S(0) \leq 0$ and $\theta(0)^\top d \in K$.

Definition 5.2. A free lunch for agent α is a net $\{(\theta^\lambda, v^\lambda); \lambda \in \Lambda\} \subset \Theta^\alpha[S] \times V$ and a bundle $k \in K$ such that

$$\theta^\lambda(1)^\top d - v^\lambda \in K \cup \{0\} \text{ for all } \lambda, \qquad v^\lambda \xrightarrow{\tau} k,$$

and

$$\liminf \theta^\lambda(0)^\top S(0) \leq 0.$$

Implicit in the definition of a free lunch is obviously a sense of continuity of agent α's preferences. We take this definition as a primitive in the analyses to follow and refer interested readers to Kreps [1981] for a host of related issues.

Here are some direct consequences of the no simple free lunch definition.

Proposition 5.1. *Suppose that a price system* S *admits no simple free lunches for some agent* α. *Then for all* $n = 1, 2, \ldots, N$, $S_n(1) = d_n$ *a.s.*

Proof. Obvious.

∎

Proposition 5.2. *Suppose that S admits no simple free lunches for agent α. There exists a linear functional $\pi_\alpha : M_\alpha \mapsto \Re$ that gives implicit prices at time zero for every marketed consumption claim (for agent α).*

Proof. It is clear that if S admits no simple free lunches for agent α, then there exists a unique implicit price for every marketed claim (for agent α). Let $\theta \in \Theta^\alpha[S]$ be a strategy that generates $m \in M_\alpha$. Define $\pi_\alpha : M_\alpha \mapsto \Re$ by $\pi_\alpha(m) = \theta(0)^\top S(0)$.

∎

Proposition 5.3. *Let $m \in M_\alpha$ be marketed for agent α. If there exist no simple free lunches, then there exists an implicit price process $\{S^\alpha_m(t); t \in \mathbf{T}\}$ for m for agent α. In addition, the implicit price process is uniquely determined up to indistinguishability.*

Proof. This is a direct consequence of no simple free lunch and the fact that there exists a portfolio of long-lived securities, available for all agents, whose implicit price process is unit throughout. (That is, hold one share of each long-lived security.) In fact, the assertion is valid in any economy where there exist ways to transfer strictly positive values across time.

∎

A corollary is immediate.

Corollary 5.1. *Suppose that agent α is better informed than agent β and that the price system admits no simple free lunches for agent α. Let $m \in M_\beta$. Then $\{S^\alpha_m(t)\}$ and $\{S^\beta_m(t)\}$ are indistinguishable processes.*

Proof. This a consequence of Theorem 4.1 and Proposition 5.3.

∎

When a price system admits no simple free lunches for a better informed agent, he cannot dynamically manufacture a consumption claim less costly than a worse informed agent can. This corollary is a complement to Theorem 4.1. The advantage of better information, when there are no simple free lunches, is *strictly* in the sense that the space of feasible net trades is bigger and *not* in the sense that some consumption claims can be manufactured less costly.

When a price system admits no free lunches we are able to say a bit more. Let Φ denote the set of nontrivial positive linear functionals on V. By the Riesz representative theorem,[4] we can identify Φ with the positive cone of $L^\infty(P)$ with the origin deleted, where $L^\infty(P)$ denotes the space of essentially bounded random variables on $(\Omega, \mathcal{F}, P)$. A *martingale measure* for agent α is a probability measure Q_α on the measurable space $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to P such that under which the price system S is a vector of $H^\alpha(S)$-martingales.

The following proposition is well known.

[4] Here we remark that since V is a Banach lattice, any positive linear functional is continuous.

Proposition 5.4. *Suppose that S admits no free lunches for agent α, then there exists $\phi_\alpha \in \Phi$ that extends π_α, denoted by $\phi_\alpha \mid M_\alpha = \pi_\alpha$.*

Proof. See Lemma 1 of Kreps [1981].

∎

The following theorem connects the no free lunch condition with martingales.

Theorem 5.1. *There exists a one-to-one correspondence between martingale measures for agent α, Q_α, and linear functionals $\phi_\alpha \in \Phi$ with $\phi_\alpha \mid M_\alpha = \pi_\alpha$. The correspondence is given by*

$$Q_\alpha(B) = \phi_\alpha(1_B) \quad \forall B \in \mathcal{F} \text{ and } \phi_\alpha(v) = E^*_\alpha(v) \quad \forall v \in V,$$

where $E^*_\alpha(\cdot)$ is the expectation under Q_α.

Proof. Let ϕ_α be an extension of π_α to all of V that lies in Φ. Thus there exists a nonzero positive essentially bounded random variable y_α on $(\Omega, \mathcal{F})$ such that $\phi_\alpha(v) = E(vy_\alpha)\ \forall v \in V$. Define a set function

$$Q_\alpha(B) = \int_B y_\alpha(\omega) P(d\omega) \qquad B \in \mathcal{F}.$$

It is easy to see that Q_α is a probability measure on $(\Omega, \mathcal{F})$. Next we want to show that S is a vector of $H^\alpha(S)$-martingales under Q_α.

Fix an integer i with $1 \le i \le N$. Let $0 \le t_1 \le t_2 \le 1$ and $\{t_1, t_2\} \subset \mathbf{T}$. For any $B \in h^\alpha_{t_1}(S)$ consider the following trading strategy:

$$\begin{aligned}
\theta_i(\omega, t) &= 1 - S_i(\omega, t_1) \text{ for } t \in (t_1, t_2] \text{ and } \omega \in B \\
&= S_i(\omega, t_2) - S_i(\omega, t_1) \text{ for } t \in (t_2, 1] \text{ and } \omega \in B \\
&= 0 \text{ otherwise,} \\
\theta_n(\omega, t) &= -S_i(\omega, t_1) \text{ for } t \in (t_1, t_2] \text{ and } \omega \in B \\
&= S_i(\omega, t_2) - S_i(\omega, t_1) \text{ for } t \in (t_2, 1] \text{ and } \omega \in B \\
&= 0 \text{ otherwise, for all } n \neq i.
\end{aligned}$$

We claim that $\theta \in \Theta^\alpha[S]$. Firstly, the stochastic integral $\theta \cdot S$ is well-defined with respect to $H^\alpha(S)$ since $B \in h^\alpha_{t_1}(S)$ and since θ is a left-continuous (therefore predictable) simple trading strategy; cf. Dellacherie and Meyer [1982, Chapter VIII]. Secondly, the fact that θ is self-financing follows from the reasoning similar to that given in Theorem 3.1 of Huang [1985a]. Thirdly, since $\mid \theta \mid$ is bounded by 1, we have

$$\begin{aligned}
E\left(\int_0^1 (\theta_n(t))^2 \, d[S_n, S_n]_t\right)^{\frac{1}{2}} &\le E\left([S_n, S_n]_1\right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

where the second inequality follows from Lemma 3.3 and Equation (3.2). Finally, we want to show that $\theta(1)^\top d \in V$. Note that $\theta(1)^\top d = 1_B\,(S_i(t_2) - S_i(t_1))$, which is bounded and naturally an element of V.

The consumption claim $1_B(S_i(t_2) - S_i(t_1))$ is marketed and has a implicit price of zero at time zero. Since t_1, t_2 and i are arbitrary, it follows that S is an N-vector of $H^\alpha(S)$-martingales under Q_α.

Conversely, let Q_α be a martingale measure for agent α. Putting $y = dQ_\alpha/dP$, y is nonzero, positive, and essentially bounded. As long as we can show that

$$E^*_\alpha\left(\int_0^1 (\theta_n(t))^2\, d[S_n, S_n]_t\right)^{\frac{1}{2}} < \infty,$$

the assertion then follows from arguments similar to those in Theorem 3.1 of Huang [1985a]. Now observe

$$\begin{aligned} E^*_\alpha\left(\int_0^1 (\theta_n(t))^2\, d[S_n, S_n]_t\right)^{\frac{1}{2}} &\leq (\operatorname{ess\,sup} y) E\left(\int_0^1 (\theta_n(t))^2\, d[S_n, S_n]_t\right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

where we have used that fact that joint variation processes are invariant under a substitution of an absolutely continuius probability measure.

∎

A corollary is immediate:

Theorem 5.1 is a generalization of the martingale results of Harrison and Kreps [1979] and Huang [1985a]. If a price system S admits no free lunches for an agent α with rational expectations, then S is a vector $H^\alpha(S)$-martingale under some probability Q_α that is absolutely continuous with respect P. The converse is also true.

We shall say that a price system S is *arbitrage-free* if it admits no free lunches for *all* agents in $\mathcal{A}$. In that case, for any agent α there exists a martingale measure Q_α. We can go on to talk about completeness of the securities markets. That being straightforward generalization of Harrision and Kreps [1979] we leave it for interested readers.

Remark 5.1: The driving force behind the martingale result is the fact that there exists a portfolio whose value is unit throughout. Thus the assumption that $\sum_{n=1}^N S_n(t) = 1$ *a.s.* for all $t \in \mathbf{T}$ is certainly not necessary. For example, if there exists a riskless asset with a zero interest rate, then a price system that admits no free lunches for, say, agent α is a $H^\alpha(S)$-martingale with respect to some martingale measure. We prepared detailed derivations of this case. They are available upon request.

6. The resolution times

In the last section we have demonstrated that an arbitrage free price system is intimately connected with martingales. In this section we shall show that no free lunches necessitates that a

better informed agent and a worse informed agent agree on the *resolution times* of a particular set of events. That is, through an arbitrage free price system some endowed asymmetry of information is equalized. We go on to formalize the linkage between *dynamically elementary market completeness*, to be defined, and a *dynamically fully revealing* price system. Under some regularity conditions a price system that admits no free lunches for a better informed agent and yields elementary completeness of markets for a worse informed agent must convey all the information that the better informed agent is endowed with to the worse informed agents over time.

Let $B \in h_1^\alpha(S)$. Define an optional time $T_B^\alpha : \Omega \mapsto [0,\infty]$ by

$$T_B^\alpha = \inf\{t \in [0,1] : E(1_B \mid h_t^\alpha(S)) = 1 \text{ or } 0\},$$

where as usual when the infimum does not exist T_B^α takes the value ∞. Note that since $B \in h_1^\alpha(S)$, we have $E(1_B \mid h_1^\alpha) = 1_B$ *a.s.* Therefore, except on a set of P-measure zero, T_B^α lies in $[0,1]$. We can always redefine T_B^α on a P-null set such that its range is $[0,1]$, however. Thus without loss of generality we shall treat T_B^α as a mapping from Ω to $[0,1]$.

Literally, T_B^α is the first time that agent α is sure that event B is to happen or not to happen with P-probability one, after observing the price system. We shall thus call T_B^α the *resolution time* of event B for agent α. Intuition suggests that if agent α is better informed than agent β, then agent α's resolution times for every $B \in h_1^\beta(S)$ will not be later than those for agent β. The following proposition formalizes this intuition. Some technical lemmas are first recorded.

Lemma 6.1. *Let $B \in h_1^\alpha(S)$. Then $B \in h_{T_B^\alpha}^\alpha(S)$.*

Proof. Define $\hat{T}_B^\alpha : \Omega \mapsto [0,\infty]$ by

$$\begin{aligned} \hat{T}_B^\alpha(\omega) &= T_B^\alpha(\omega) \qquad \forall \omega \in B \\ &= \infty \qquad \forall \omega \notin B. \end{aligned}$$

It is clear that

$$\hat{T}_B^\alpha = \inf\{t \in [0,1] : E(1_B \mid h_t^\alpha(S)) = 1\}.$$

Thus $\hat{T}_B^\alpha$ is an $H^\alpha(S)$-optional random variable. It then follows from Theorem IV.53 of Dellacherie and Meyer [1982] that $B \in h_{T_B^\alpha}^\alpha(S)$.

∎

Remark 6.1: For the definitions of optional random variables and $h_{T_B^\alpha}^\alpha(S)$, see Chung and Williams [1983, Section 1.7].

Lemma 6.2. *If $H^\beta(S) \subset H^\alpha(S)$, then any $H^\beta(S)$-optional random variable is $H^\alpha(S)$-optional. Furthermore, let T be $H^\beta(S)$-optional. Then $h_T^\beta(S) \subset h_T^\alpha(S)$.*

Proof. Let T be $H^\beta(S)$-optional. Then we have

$$\{T \le t\} \in h_t^\beta(S) \quad \forall t \in \mathbf{T}$$

by the definition of optionality. Since $h_t^\beta(S) \subset h_t^\alpha(S)\ \forall t \in \mathbf{T}$, we know

$$\{T \leq t\} \in h_t^\alpha(S) \quad \forall t \in \mathbf{T}.$$

This, by the definition of optionality, implies that T is $H^\alpha(S)$-optional.

Next we note that $h_T^\beta(S)$ contains all the sets $B \in h_1^\beta(S)$ such that $B \bigcap \{T \leq t\} \in h_t^\beta(S)$ $\forall t \in \mathbf{T}$. Let $B \in h_T^\beta(S)$. Since $H^\beta(S) \subset H^\alpha(S)$, we know $B \in h_1^\alpha(S)$ and $B \bigcap \{T \leq t\} \in h_t^\alpha(S)$ $\forall t \in \mathbf{T}$. Thus $B \in h_T^\alpha(S)$.

∎

Proposition 6.1. *Suppose that* $\mathbf{F}^\beta \subset \mathbf{F}^\alpha$. *Let* $B \in h_1^\beta(S)$. *Then* $T_B^\alpha \leq T_B^\beta$ P-a.s.

Proof. From the hypothesis, it is claer that $H^\beta(S) \subset H^\alpha(S)$. From Lemma 6.1 we know that $B \in h^\beta_{T_B^\beta}(S)$. Thus

$$E(1_B \mid h^\beta_{T_B^\beta}(S)) = 1_B \quad P - a.s.$$

From Lemma 6.2 we know that $h^\beta_{T_B^\beta}(S) \subset h^\alpha_{T_B^\beta}(S)$. Therefore by iterative expectation we get

$$\begin{aligned} E\left(1_B \mid h^\beta_{T_B^\beta}(S)\right) &= E\left(E\left(1_B \mid h^\alpha_{T_B^\beta}(S)\right) \mid h^\beta_{T_B^\beta}(S)\right) \\ &= 1_B \qquad P - a.s. \end{aligned}$$

Since $E(1_B \mid h^\alpha_{T_B^\beta}(S))$ lies in $[0,1]$ with P-probability one, we must have

$$E\left(1_B \mid h^\alpha_{T_B^\beta}(S)\right) = 1_B \qquad P - a.s.,$$

which implies that $T_B^\alpha(S) \leq T_B^\beta(S)$ P-a.s.

∎

The following proposition shows that the definition of resolution times is invariant under a substitution of an equivalent probability measure.

Proposition 6.2. *Let* $B \in h_1^\alpha(S)$, *let* Q *be a probability measure equivalent to* P, *and let* T_B^α *and* $\widehat{T}_B^\alpha$ *be resolution times for* B *with respect to* P *and* Q, *respectively. Then* $T_B^\alpha = \widehat{T}_B^\alpha$ P-a.s. *and therefore* Q-a.s.

Proof. Putting $\xi = dQ/dP$ and fixing a right continuous version $\xi(t) = E(\xi \mid h_t^\alpha(S))$. Since Q is absolutely continuous with respect to P, $\{\xi(t)\}$ is a strictly positive process except possibly on a P-null set. We have

$$\begin{aligned} E_Q(1_B \mid h^\alpha_{T_B^\alpha}) &= \frac{E(1_B \xi \mid h^\alpha_{T_B^\alpha}(S))}{\xi(T_B^\alpha)} \\ &= 1_B \qquad P - a.s. \end{aligned}$$

This implies that $\widehat{T}_B^\alpha \leq T_B^\alpha$ P-a.s. and therefore Q-a.s. since Q is absolutely continuous with respect to P.

By the hypothesis that Q and P are in fact equivalent, we know P is absolutely continuous with respect to Q. Thus we can reverse the above argument and have $T_B^\alpha \geq \widehat{T}_B^\alpha$ P-a.s. and Q-a.s. Hence we must have $T_B^\alpha = \widehat{T}_B^\alpha$ P-a.s. and Q-a.s.

∎

Before proceeding to the main theorems of this section, we first give some strengthings of the results of Section 5.

We shall assmue henceforth that the probability space $(\Omega, \mathcal{F}, P)$ is separable. Let Ψ denote the space of strictly positive linear functionals on V. An *equivalent martingale measure* for agent α is a martingale measure for him that is equivalent to P.

Let Ψ denote the space of strictly positive linear functionals on V. The following proposition follows from an extension of Theorem 3 of kreps [1981], whose proof is given in the Appendix.

Proposition 6.3. *Suppose that S admits no free lunches for agent α. Then there exists $\psi_\alpha \in \Psi$ such that $\psi_\alpha \mid M_\alpha = \pi_\alpha$.*

Proof. Given that $(\Omega, \mathcal{F}, P)$ is a separable probability space, V is a separable normed space. The assertion then follows from the theorem in the Appendix.

∎

A direct consequence of the above proposition and Theorem 5.1 is:

Proposition 6.4. *There exists a one-to-one correspondence between equivalent martingale measures for agent α, Q_α, and linear functionals $\psi_\alpha \in \Psi$ with $\psi_\alpha \mid M_\alpha = \pi_\alpha$. The correspondence is given by*

$$Q_\alpha(B) = \psi_\alpha(1_B) \quad \forall B \in \mathcal{F} \text{ and } \psi_\alpha(v) = E_\alpha^*(v) \quad \forall v \in V,$$

where $E_\alpha^*(\cdot)$ denotes the expectation under Q_α.

When the probability space is separable, no free lunches for agent α implies the existence of a strictly positive extension of π_α to all of V. This in turn implies the existence of an equivalent martingale measure for agent α.

Since all the probability measures to appear will be equivalent probability measures, we shall henceforth simply use *a.s.* to denote *almost surely* under any probability measure involved.

Here is our first main result of this section.

Theorem 6.1. *Suppose that agent α is better informed than agent β, and that the price system admits no free lunches for agent α. Let $B \in h_1^\beta(S)$ be such that $1_B \in M_\beta$. Then*

$$T_B^\alpha = T_B^\beta \qquad P - a.s.$$

Proof. Let $\{S_B^\alpha(t)\}$ and $\{S_B^\beta(t)\}$ denote the implicit price processes for 1_B for agents α and β, respectively. From Corollary 5.1 we know that $\{S_B^\alpha(t)\}$ and $\{S_B^\beta(t)\}$ are indistinguishable processes. Therefore, from now on we use $\{S_B(t)\}$ to denote both.

From Propositions 5.1, 6.3, and 6.4 we know that

$$\begin{aligned} S_B(t) &= E_\alpha^*(1_B \mid h_t^\alpha(S)) \quad \forall t \in \mathbf{T} \quad a.s. \\ &= E_\beta^*(1_B \mid h_t^\beta(S)) \quad \forall t \in \mathbf{T} \quad a.s., \end{aligned}$$

where as usual $E_\alpha^*(\cdot)$ and $E_\beta^*(\cdot)$ denote the expectations under equivalent martingale measures Q_α and Q_β for agents α and β, respectively. Thus the resolution times of B under $(H^\alpha(S), Q_\alpha)$ and under $(H^\beta(S), Q_\beta)$ are equal almost surely. It then follows from Proposition 6.2 that the resolution times of B under $(H^\alpha(S), P)$ and under $(H^\beta(S), P)$ are equal almost surely.

∎

Commonly marketed events will be revealed to differentially informed agents in the same time. If the resolution time of B under $(\mathbf{F}^\alpha, P)$ is strictly earlier than that under $(\mathbf{F}^\beta, P)$, the above theorem implies that the price system that admits no free lunches for agent α must carry *enough* information to *equalize* the assymetry of information about B between agents α and β.

We shall say that markets are *elementarily complete* for agent α if $1_B \in M_\alpha$ for all $B \in h_1^\alpha(S)$. A price system S is said to be *dynamically fully revealing* between agents α and β if $h_t^\alpha(S) = h_t^\beta(S)$ $\forall t \in \mathbf{T}$.

Here is our second main result of this section.

Theorem 6.2. *Suppose that agent α is better informed than agent β and that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$. If the markets are elementarily complete for agent β, then a price system that admits no free lunches for agent α must be dynamically fully revealing between α and to β.*

Proof. The hypothesis that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$ implies that $h_1^\alpha(S) = h_1^\beta(S)$. The elementary market completeness hypothesis implies that any event in $h_1^\alpha(S)$ is marketed. It then follows from Theorem 6.1 that the resolution times for all the events in $h_1^\alpha(S)$ under $(H^\alpha(S), P)$ and under $(H^\beta(S), P)$ are equal. Now we claim that $h_t^\alpha(S) = h_t^\beta(S)$ $\forall t \in \mathbf{T}$. Suppose this is not the case. Then there exists $t \in \mathbf{T}$ such that $h_t^\beta(S) \subset h_t^\alpha(S)$ and $h_t^\beta(S) \neq h_t^\alpha(S)$. Let $B \in h_t^\alpha(S)$ and $B \notin h_t^\beta(S)$. We know $P(B) > 0$ since all the filtrations are completed. Also, we have

$$E(1_B \mid h_t^\alpha(S)) = 1_B \qquad a.s.$$

and on a set of strictly positive P-measure

$$E(1_B \mid h_t^\beta(S)) \neq 1_B.$$

By the right-continuity of the filtrations, we know the resolution times for B under $H^\alpha(S)$ and under $H^\beta(S)$ are not equal, a contradiction.

∎

A corollary is immediate:

Corollary 6.1. *Suppose that agent β is the least informed agents in $\mathcal{A}$, that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$ for all agent α in $\mathcal{A}$, and that markets are elementarily complete for agent β. Then any arbitrage free price system must be dynamically fully revealing in the sense that*

$$h_t^\beta(S) = h_t^\alpha(S) \qquad \forall t \in \mathbf{T} \text{ and all agents in } \mathcal{A}.$$

This section, in our view, is the most important contribution of this paper. It has been recognized by probability theorists that the way information is revealed over time is closely related to the behaviors of martingales and optional times. In an economic context, the former has been extensively studied by Huang [1985a]; while the latter is introduced in this section. The theorem proved in the Appendix and used to demonstrate Proposition 6.3 is of independent interest. The conditions in Theorem 3 of Kreps [1981] to insure that there exists a strictly positive continuous extension of a linear functional are sometimes hard to verify in applications. It turns out that as along as we have separability matters are simple.

7. An example

In this section we present an example along the lines of Remark 5.1. There are two long-lived securities traded, one risky and one riskless with a zero interest rate. Agent α gets to observe a standard Brownian motion as it comes along. Agent β knows at time zero the value of the Brownian motion at time one but cannot observe its sample paths over time. Two agents are differentially informed and there is no ordering between the two endowed information structures. Markets are elementarily complete for both agents and the price system is dynamically fully revealing between them.

Formally, there is defined on the basic probability space $(\Omega, \mathcal{F}, P)$ a standard Brownian motion, denoted by $W = \{W(t); t \in [0,1]\}$. We denote by $\mathbf{F}^W = \{\mathcal{F}_t^W; t \in [0,1]\}$ the natural filtration of W. The set of trading dates of the economy $\mathbf{T}$ is taken to be $[0,1]$.

We assume that $\mathbf{F}^\alpha = \mathbf{F}^W$ and that, for all $t \in \mathbf{T}$, $\mathcal{F}_t^\beta = \sigma\{W(1)\}$, the tribe generated by $W(1)$. Note that although consumption only occurs at time 1, knowing the value of the Brownian motion at time 1 at the very beginning agent β is not better informed. The optimal net trade for agent β, if he could observe W directly, might well be path-dependent.

There are two long-lived securities traded. Their payoff structures are

$$d_1(\omega) = 1_\Omega(\omega)$$

$$d_2(\omega) = 2W(\omega,1) - \int_0^1 \left(\gamma + \frac{W(\omega,1) - W(\omega,s)}{1-s}\right) ds,$$

where γ is a real number. The price system is

$$S_1(\omega,t) = 1_\Omega(\omega)$$
$$S_2(\omega,t) = W(\omega,1) + W(\omega,t) - \int_0^t \left(\gamma + \frac{W(\omega,1) - W(\omega,s)}{1-s}\right) ds.$$

Corollary 1.1 of Jeulin and Yor [1979] implies that

$$W(t) + (1-t)W(1) = W(1) + (1-t)\left(\int_0^t \frac{1}{1-s} dS_2(s) + \int_0^t \frac{\gamma}{1-s} ds\right) \quad t \in [0,1] \quad P-a.s.$$

Since γ is a constant, this implies that $\{W(t) + (1-t)W(1); t \in [0,1]\}$ and $\{S_2(t); t \in [0,1]\}$ generate the same filtration, denoted henceforth by $\mathbf{F}^S = \{\mathcal{F}_t^S; t \in \mathbf{T}\}$. We assume that $\mathbf{F}^S$ has been completed. Corollary 1.1(a) of Jeulin and Yor implies that

$$\mathcal{F}_t^S = \bigwedge_{\epsilon>0} \left\{\mathcal{F}_{t+\epsilon}^W \bigvee \sigma\{W(1)\}\right\}.$$

Thus $\mathbf{F}^S$ is right-continuous and is finer than either $\mathbf{F}^\alpha$ or $\mathbf{F}^\beta$. It follows then that $H^\alpha(S) = H^\beta(S) = \mathbf{F}^S$. That is, the price system is dynamically fully revealing between α and β.

Next we claim that S admits no free lunches. It suffices to demonstrate the existence of a martingale measure. Put

$$\mathcal{W}(t) = W(t) - \int_0^t \frac{W(1) - W(s)}{1-s} ds.$$

Theorem 1 of Jeulin and Yor shows that $\{\mathcal{W}(t); t \in \mathbf{T}\}$ is a standard Brownian motion with respect to $\mathbf{F}^S$ under P. Now define

$$\xi = \exp\{\gamma \mathcal{W}(1) - \frac{1}{2}\gamma^2\},$$

and a set function

$$Q(B) = \int_B \xi(\omega) P(d\omega) \quad B \in \mathcal{F}_1^S.$$

Theorem 6.1 and lemma 6.5 of Liptser and Shiryayev [1977] ensures that Q is a probability measure on $(\Omega, \mathcal{F}_1^S)$ equivalent to P. It then follows from the Girsanov theorem (cf. Liptser and Shiryayev [1977, pp. 225]) that

$$\mathcal{W}^*(t) = \mathcal{W}(t) - \gamma t = W(1) - \int_0^t \left(\gamma + \frac{W(1) - W(s)}{1-s}\right) ds \quad t \in \mathbf{T}$$

is a standard Brownian motion adapted to $\mathbf{F}^S$ under Q. Thus S_2 is a $\mathbf{F}^S$-martingale under Q, since $S_2(t) = \mathcal{W}^*(t) + W(1)$.

Finally, the fact that markets are elementarily complete for both agents follows from Corollary 1.1(e) of Jeulin and Yor [1979].

8. Discussions, generalizations, and concluding remarks

The assumption that the price system has been normalized cannot be taken lightly. In a Walrasian economy, a normalization of prices is economically neutral. In a rational expectations economy such a procedure will not be economically neutral in general. The information content of a price system may be altered by changing numeraires. Our analysis can be generalized in the following direction, however. Take S to be a vector of F-semimartingales satisfying all the regularity conditions except that they sum to a process bounded away from zero. We can now normalize prices to sum to one, denoted by S^*, but still let agents have access to the information generated by S. Then all the characterizations in this paper apply to S^*.

The assumption that consumption only occurs at date one is made for the ease of exposition. All the results before Section 6 are readily extended to an economy where the commodity space is the space of bounded variation processes representing agents' accumulated net trades; cf. Huang [1985b]. Results on resolution times that need an equivalent martingale measure cannot be treated in the general case without further assumption. The reason is that the commodity space is no longer normable. The condition needed is that a price system be *viable*; cf. Kreps [1981].

Other than the criticism in the first paragraph, the results presented in this paper are robust. No arbitrage is a weak requirement of a price system. We have demonstrated that this requirement has pervasive implications. In a common information economy, it is the connection between martingales and an arbitrage free price system observed by Harrison and Kreps [1979] that makes a dynamic equilibrium theory possible; cf. Duffie and Huang [1985]. In a differential information context, no arbitrage is the minimum condition a rational expectations equilibrium price system must satisfy. It necessitates that a price system be a martingale under some probability. Readers should convince themselves that putting this papar and Duffie and Huang [1985] together it is not hard to prove the exietence of a dynamic rational expectations equilibrium with a fully revealing price system. Whether there exists a partially revealing dynamic rational expectations equilibrium is an open question.

Appendix

Let X be a separable normed space and let K be a cone in X with the origin deleted. Let x_1 and x_2 be elements of X. We shall write $x_1 \geq x_2$ if $x_1 - x_2 \in K \bigcup \{0\}$. A securities markets model is a pair (M, π), a subspace M of X and a linear functional π on M. A free lunch in the securities markets model is a sequence $\{(m_n, x_n); n = 1, 2, \ldots\} \subset M \times X$ such that

$$m_n \geq x_n \qquad x_n \to k \in K$$

$$\text{and } \liminf_n \pi(m_n) \leq 0.$$

Theorem A.1. *Suppose that (M, π) admits no free lunches. Then there exists a $\psi \in \Psi$ such that $\psi \mid M = \pi$, where Ψ is the space of strictly positive continuous linear functionals on X.*

Proof. From Lemma 5 of Kreps [1981] we know that there exists a collection of equicontinuous positive continuous linear functionals on X: $\Gamma = \{\psi_k; k \in K\}$ such that $\psi_k(k) > 0$ and $\psi_k \mid M = \pi$. It follows that Γ is a separable metric space in the relative weak* topology; cf. Theorem 5.4.7 of Schaefer [1980]. Let $\{\psi_{k_n}; n = 1, 2, \ldots\}$ be a countable dense subset of Γ in the relative weak* topology and let $\{\lambda_n; n = 1, 2, \ldots\}$ be a sequence of strictly positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Now define

$$\psi = \sum_{n=1}^{\infty} \lambda_n \psi_{k_n}.$$

It is quickly verified that ψ is a strictly positive continuous linear functional on X with $\psi \mid M = \pi$. ∎

References

1 K. Chung and R. Williams, *An Introduction to Stochastic Integration*, Birkhauser Boston Inc., 1983.

2 C. Dellacherie and P. Meyer, *Probabilities and Potential A: General Theory of Process*, North-Holland Publishing Company, New York, 1978.

3 C. Dellacherie and P. Meyer, *Probabilities and Potential B: Theory of Martingales*, North-Holland Publishing Company, New York, 1982.

4 D. Duffie and C. Huang, Implementing Arrow-Debreu equilibria by continuous trading of few long-linved securities, forthcoming in **Econometrica**, 1985.

5 D. Duffie, Stochastic equilibria: existence, spanning number, and the 'no expected gains from trade' hypothesis, Research Paper No. 762, Graduate School of Business, Stanford University, 1984.

6 M. Harrison and D. Kreps, Martingales and arbitrage in multiperiod securities markets, **J. Econ. Theor.** 20 (1979), 381-408.

7 C. Huang, Information structure and equilibrium asset prices, **J. Econ. Theor.** forthcoming 1985.

8 C. Huang, Information structure and viable price systems, MIT mimeo, 1985.

9 J. Jacod, *Calcul Stochastique et Problemes de Martingales*, Lecture Notes in Mathematics 714, Springer-Verlag, New York, 1979.

10 T. Jeulin and M. Yor, Inegalites de Hardy, semi-martingales et faux amis, Seminaire de Probabilites XIII, L.N. in Math. 721, 332-359, 1979.

11 D. Kreps, A note on fulfilled expectations equilibrium, **Journal of Economic Theory**, 1977.

12 D. Kreps, Arbitrage and equilibrium in economies with infinitely many commodities, **J. Math. Econ.** 8 (1981), 15-35.

13 R. Liptser and A. Shiryayev, *Statistics of Random Processes I: General Theory*, Springer-Verlag, New York, 1977.

14 H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York, 1980.

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